Bonus 2: Solution

If we define the following function:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational;} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Then we claim that $f(x)$ is not cts at any real number. Indeed, let a be an arbitrary real number. In any interval around a, there are both rational numbers and irrational numbers. So in any interval around a, there will be a rational number x, so $f(x) = 0$ and an irrational y, so $f(y) = 1$. So no matter how close we get to a, we will get both $f(x) = 0$ and $f(x) = 1$. Thus the limit of $f(x)$ as x approaches a cannot exist, since $f(x)$ does not approach a consistent value as x gets close to a.

Thus since

$$\lim_{x \to a} f(x)$$

does not exist, $f(x)$ is not cts at a. Since a was arbitrary, $f(x)$ is not cts at any real number.

It is important to give a reason, as above, why $f(x)$ is not cts at any real number. Similar-looking functions may be cts; for example, one can show that the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational;} \\ x & \text{if } x \text{ is irrational.} \end{cases}$$

is cts at 0.